Optimal Power Flow using Interior Point Method

EE 8725

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Outline

Interior Point Method (Primal Dual)

Optimal Power Flow

Intro to IPM

Minimize: f(x)

Subject to: g(x) = 0,

h(x) ≤0,

where f(x), x, g(x), h(x), represent, respectively, the objective function, the decision variables, the equality constraints and the inequality constraints.

x = [x_1, x_2, x_3,….., x_{nv}]T

Steps in IPM

1. Log Barrier
2. Necessary Conditions (KKT)
3. Newton Direction
4. Updating variables
5. Convergence check

Log Barrier Method

$$\begin{aligned} \text{Minimize:} \quad & f(\boldsymbol{x}) \\ \text{Subject to:} \quad & \boldsymbol{g}(\boldsymbol{x}) = \boldsymbol{0}, \\ & \boldsymbol{h}(\boldsymbol{x}) + \boldsymbol{s} = \boldsymbol{0}, \\ & \boldsymbol{s} \geq \boldsymbol{0}, \end{aligned}$$

where,

$$\boldsymbol{s} = [s_1, s_2, \ldots, s_{n_c}]^T$$

$$\begin{aligned} \text{Minimize:} \quad & f_\mu = f(\boldsymbol{x}) - \mu^k \sum_{i=1}^{n_c} \ln(s_i) \\ \text{Subject to:} \quad & \boldsymbol{g}(\boldsymbol{x}) = \boldsymbol{0}, \\ & \boldsymbol{h}(\boldsymbol{x}) + \boldsymbol{s} = \boldsymbol{0}, \\ & \boldsymbol{s} \geq \boldsymbol{0}, \end{aligned}$$

According to Fiacco and McCormick theorem, as $\mu_k \rightarrow 0$ usually a local optimal solution is reached.

$$\boldsymbol{L}_\mu = f(\boldsymbol{x}) - \mu^k \sum_{i=1}^{n_c} \ln(s_i) - \boldsymbol{\lambda}^T \Big[-\boldsymbol{g}(\boldsymbol{x}) \Big] - \boldsymbol{\gamma}^T \Big[-\boldsymbol{h}(\boldsymbol{x}) - \boldsymbol{s} \Big]$$

Necessary optimality conditions

The first order necessary optimality conditions are obtained by setting to zero the derivatives of L μ with respect x, λ, γ and s:

$$\nabla_x \boldsymbol{L}_\mu = \nabla_x f(\boldsymbol{x}) + \nabla_x \boldsymbol{g}(\boldsymbol{x}) \cdot \boldsymbol{\lambda} + \nabla_x \boldsymbol{h}(\boldsymbol{x}) \cdot \boldsymbol{\gamma} = \mathbf{0} \longrightarrow \text{Eq.1}$$

$$\nabla_\lambda \boldsymbol{L}_\mu = \boldsymbol{g}(\boldsymbol{x}) = \mathbf{0} \longrightarrow \text{Eq.2}$$

$$\nabla_\gamma \boldsymbol{L}_\mu = \boldsymbol{h}(\boldsymbol{x}) + \boldsymbol{s} = \mathbf{0} \longrightarrow \text{Eq.3}$$

$$\nabla_s \boldsymbol{L}_\mu = -\mu^k \cdot \boldsymbol{S}^{-1} \cdot \boldsymbol{e} + \boldsymbol{\gamma} = \mathbf{0} \longrightarrow \text{Eq.4}$$

KKT Interpretation: Eq. 2 and Eq. 3, along with s≥0, ensures primal feasibility
Eq. 1, together with γ ≥ 0,ensure dual feasibility.
Eq. 4 is the perturbed complementary condition.

First order derivatives

$$\nabla_x f(\boldsymbol{x}) = \begin{bmatrix} \dfrac{\partial f}{\partial x_1} & \dfrac{\partial f}{\partial x_2} & \cdots & \dfrac{\partial f}{\partial x_{n_v}} \end{bmatrix}^T$$

$$\nabla_x \boldsymbol{g}(\boldsymbol{x}) = \begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_2}{\partial x_1} & \cdots & \frac{\partial g_{n_{ceq}}}{\partial x_1} \\ \frac{\partial g_1}{\partial x_2} & \frac{\partial g_2}{\partial x_2} & \cdots & \frac{\partial g_{n_{ceq}}}{\partial x_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial g_1}{\partial x_{n_v}} & \frac{\partial g_2}{\partial x_{n_v}} & \cdots & \frac{\partial g_{n_{ceq}}}{\partial x_{n_v}} \end{bmatrix}$$

$$\nabla_x \boldsymbol{h}(\boldsymbol{x}) = \begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \frac{\partial h_2}{\partial x_1} & \cdots & \frac{\partial h_{nc}}{\partial x_1} \\ \frac{\partial h_1}{\partial x_2} & \frac{\partial h_2}{\partial x_2} & \cdots & \frac{\partial h_{nc}}{\partial x_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial h_1}{\partial x_{n_v}} & \frac{\partial h_2}{\partial x_{n_v}} & \cdots & \frac{\partial h_{nc}}{\partial x_{n_v}} \end{bmatrix}$$

Newton Direction

$$\begin{bmatrix} \nabla_x^2 L_\mu & \nabla_x g(x) & \nabla_x h(x) & 0 \\ \nabla_x g(x)^T & 0 & 0 & 0 \\ \nabla_x h(x)^T & 0 & 0 & I \\ 0 & 0 & I & \nabla_s^2 L_\mu \end{bmatrix} \cdot \begin{bmatrix} \Delta x \\ \Delta \lambda \\ \Delta \gamma \\ \Delta s \end{bmatrix} = - \begin{bmatrix} \nabla_x L_\mu \\ \nabla_\lambda L_\mu \\ \nabla_\gamma L_\mu \\ \nabla_s L_\mu \end{bmatrix}$$

- This involves computing the step size in the Newton direction, updating the variables, and reducing the parameter μ.
- The algorithm terminates when primal and dual infeasibilities and the complementarity gap fall bellow predetermined tolerances.
- The Newton direction is used as a means to follow the path of minimization which is parameterized by μ.

Hessian

$$\nabla_x^2 \boldsymbol{L}_\mu = \nabla_x^2 f(\boldsymbol{x}) + \sum_{i=1}^{n_{ceq}} \lambda_i \cdot \nabla_x^2 g_i(\boldsymbol{x}) + \sum_{j=1}^{n_c} \gamma_j \cdot \nabla_x^2 h_j(\boldsymbol{x})$$

$$\nabla_x^2 f(\boldsymbol{x}) = \boldsymbol{H}_f = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_{n_v}} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_{n_v}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_{n_v} \partial x_1} & \frac{\partial^2 f}{\partial x_{n_v} \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_{n_v}^2} \end{bmatrix}$$

$$\nabla_x^2 g_i(\boldsymbol{x}) = \boldsymbol{H}_{g_i} = \begin{bmatrix} \frac{\partial^2 g_i}{\partial x_1^2} & \frac{\partial^2 g_i}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 g_i}{\partial x_1 \partial x_{n_v}} \\ \frac{\partial^2 g_i}{\partial x_2 \partial x_1} & \frac{\partial^2 g_i}{\partial x_2^2} & \cdots & \frac{\partial^2 g_i}{\partial x_2 \partial x_{n_v}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 g_i}{\partial x_{n_v} \partial x_1} & \frac{\partial^2 g_i}{\partial x_{n_v} \partial x_2} & \cdots & \frac{\partial^2 g_i}{\partial x_{n_v}^2} \end{bmatrix}$$

$$\nabla_x^2 h_i(\boldsymbol{x}) = \boldsymbol{H}_{h_i} = \begin{bmatrix} \frac{\partial^2 h_i}{\partial x_1^2} & \frac{\partial^2 h_i}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 h_i}{\partial x_1 \partial x_{n_v}} \\ \frac{\partial^2 h_i}{\partial x_2 \partial x_1} & \frac{\partial^2 h_i}{\partial x_2^2} & \cdots & \frac{\partial^2 h_i}{\partial x_2 \partial x_{n_v}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 h_i}{\partial x_{n_v} \partial x_1} & \frac{\partial^2 h_i}{\partial x_{n_v} \partial x_2} & \cdots & \frac{\partial^2 h_i}{\partial x_{n_v}^2} \end{bmatrix}$$

Updating Variables

$$\boldsymbol{x}^{k+1} = \boldsymbol{x}^{k} + k_s \cdot \alpha_P \cdot \Delta\boldsymbol{x}$$

$$\boldsymbol{s}^{k+1} = \boldsymbol{s}^{k} + k_s \cdot \alpha_P \cdot \Delta\boldsymbol{s}$$

$$\boldsymbol{\lambda}^{k+1} = \boldsymbol{\lambda}^{k} + k_s \cdot \alpha_D \cdot \Delta\boldsymbol{\lambda}$$

$$\boldsymbol{\gamma}^{k+1} = \boldsymbol{\gamma}^{k} + k_s \cdot \alpha_D \cdot \Delta\boldsymbol{\gamma}$$

- Scalars α_P and α_D are the step length parameters.
- Scalar k_s ∈(0, 1) represent the safety factor that guarantees the strict positive condition of the slack variables s and γ during each iteration.

- The value of k_s can be updated during each iteration as needed.

Primal and dual step length

To update the variables, it is necessary to know the size of the primal and dual steps. These steps must be calculated in such a way that they guarantee the positive condition of the variables s and γ

$$\alpha_{PMAX} = \min\left[\min_{\Delta s_i<0} \frac{s_i}{|\Delta s_i|}, 1\right]$$

$$\alpha_{DMAX} = \min\left[\min_{\Delta \gamma_i<0} \frac{\gamma_i}{|\Delta \gamma_i|}, 1\right]$$

The length of the primal and dual steps can be chosen from the intervals $\alpha_P \in (0, \alpha_{PMAX}]$ and $\alpha_D \in (0, \alpha_{DMAX}]$.

Update μ

In order to calculate μk, the complementarity gap is computed at each iteration k by:

$$\rho^k = \left(\boldsymbol{\gamma}^k\right)^T \cdot \boldsymbol{s}^k$$

In each iteration k the value of μk can be computed based on the decrement of the complementarity gap

$$\mu^k = \sigma \cdot \frac{\rho^k}{n_c}$$

σ is the centering parameter, usually σ = 0.1

Convergence check

$$
\begin{aligned}
v_1^k &= \max\left[\max\left(\boldsymbol{h}(\boldsymbol{x}^k)\right), \|\boldsymbol{g}(\boldsymbol{x}^k)\|_\infty\right] \\
v_2^k &= \frac{\|\nabla_x f(\boldsymbol{x}^k) + \nabla_x \boldsymbol{g}(\boldsymbol{x}^k)\cdot\boldsymbol{\lambda}^k + \nabla_x \boldsymbol{h}(\boldsymbol{x}^k)\cdot\boldsymbol{\gamma}^k\|_\infty}{1 + \|\boldsymbol{x}^k\|_2} \\
v_3^k &= \frac{\rho^k}{1 + \|\boldsymbol{x}^k\|_2} \\
v_4^k &= \mu^k
\end{aligned}
$$

Minimization of Cost Function

Rectangular:

$$\boldsymbol{x} = \left[V_2^{Re}, V_2^{Im}, V_3^{Re}, V_3^{Im}, \ldots, V_N^{Re}, V_N^{Im}, P_{G1}, P_{G2}, \ldots, P_{Gn_g}, Q_{G1}, Q_{G2}, \ldots, Q_{Gn_g}\right]^T$$

1. Load constraints where P_{Lk} and Q_{Lk} are fixed:

$$\sum_{m=1}^{N}\left[Y_{km}^{Re}\left(V_m^{Re}V_k^{Re} + V_m^{Im}V_k^{Im}\right) + Y_{km}^{Im}\left(V_m^{Re}V_k^{Im} - V_m^{Im}V_k^{Re}\right)\right] + P_{Lk} = 0$$

$$\sum_{m=1}^{N}\left[Y_{km}^{Re}\left(V_m^{Re}V_k^{Im} - V_m^{Im}V_k^{Re}\right) - Y_{km}^{Im}\left(V_m^{Re}V_k^{Re} + V_m^{Im}V_k^{Im}\right)\right] + Q_{Lk} = 0$$

2. Generation constraints where P_{Gk} and Q_{Gk} are not fixed:

$$\sum_{m=1}^{N}\left[Y_{km}^{Re}\left(V_m^{Re}V_k^{Re} + V_m^{Im}V_k^{Im}\right) + Y_{km}^{Im}\left(V_m^{Re}V_k^{Im} - V_m^{Im}V_k^{Re}\right)\right] - P_{Gk} = 0$$

$$\sum_{m=1}^{N}\left[Y_{km}^{Re}\left(V_m^{Re}V_k^{Im} - V_m^{Im}V_k^{Re}\right) - Y_{km}^{Im}\left(V_m^{Re}V_k^{Re} + V_m^{Im}V_k^{Im}\right)\right] - Q_{Gk} = 0$$

3. Voltage magnitude equality constraints:

$$|V_{Gk}|^2 = \left(V_k^{Re}\right)^2 + \left(V_k^{Im}\right)^2$$

Polar:

$$\boldsymbol{x} = \left[\delta_2, \delta_3, \ldots, \delta_N, |V_2|, |V_3|, \ldots, |V_N|, P_{G1}, P_{G2}, \ldots, P_{Gn_g}, Q_{G1}, Q_{G2}, \ldots, Q_{Gn_g}\right]^T$$

1. Load constraints where P_{Lk} and Q_{Lk} are fixed:

$$\sum_{m=1}^{N}\left[|V_k|\sum_{m=1}^{N}|Y_{km}||V_m|\cos(\psi_{km})\right] + P_{Lk} = 0$$

$$\sum_{m=1}^{N}\left[|V_k|\sum_{m=1}^{N}|Y_{km}||V_m|\sin(\psi_{km})\right] + Q_{Lk} = 0$$

2. Generation constraints where P_{Gk} and Q_{Gk} are not fixed:

$$\sum_{m=1}^{N}\left[|V_k|\sum_{m=1}^{N}|Y_{km}||V_m|\cos(\psi_{km})\right] - P_{Gk} = 0$$

$$\sum_{m=1}^{N}\left[|V_k|\sum_{m=1}^{N}|Y_{km}||V_m|\sin(\psi_{km})\right] - Q_{Gk} = 0$$

3. Voltage magnitude equality constraints:

$$|V_{Gk}| = Constant$$

4. Voltage magnitude inequality constraints:

$$|V_{Lk}|^{Min} \leq |V_{Lk}| \leq |V_{Lk}|^{Max}$$

5. Generation loading constraints:

$$P_{Gk}^{Min} \leq P_{Gk} \leq P_{Gk}^{Max}$$

$$Q_{Gk}^{Min} \leq Q_{Gk} \leq Q_{Gk}^{Max}$$

6. Line flow constraints:

$$P_{km}^{Min} \leq P_{km} \leq P_{km}^{Max}$$

Grad and Hessian of f(x)

$$\nabla_x f(\boldsymbol{x}) = \left[\ \mathbf{0} \left| \ \frac{\partial f}{\partial P_{G1}} \quad \frac{\partial f}{\partial P_{G2}} \quad \cdots \quad \frac{\partial f}{\partial P_{Gn_g}} \ \right| \mathbf{0}\ \right]^T$$

Very Sparse

$$\nabla_x^2 f(\boldsymbol{x}) = \boldsymbol{H}_f = \left[\begin{array}{c|cccc|c} \underbrace{\mathbf{0}}_{u\times u} & & & & & \underbrace{\mathbf{0}}_{u\times n_g} \\ \hline & \frac{\partial^2 f}{\partial P_{G1}^2} & & & & \\ \underbrace{\mathbf{0}}_{n_g\times u} & & \frac{\partial^2 f}{\partial P_{G2}^2} & & & \underbrace{\mathbf{0}}_{n_g\times n_g} \\ & & & \ddots & & \\ & & & & \frac{\partial^2 f}{\partial P_{Gn_g}^2} & \\ \hline \underbrace{\mathbf{0}}_{n_g\times u} & & & & & \underbrace{\mathbf{0}}_{n_g\times n_g} \end{array}\right]$$

Extremely Sparse

Grad and Hessian of g(x)

All the equality constraints are stored as

$$\boldsymbol{g}(\boldsymbol{x}) = [P_1, P_2, \dots, P_N, Q_1, Q_2, \dots, Q_N, |V_2|, |V_3|, \dots, |V_N|]^T$$

Gradient:

$$\nabla_x \boldsymbol{g}(\boldsymbol{x}) = \begin{bmatrix} [\nabla_x P_k] & [\nabla_x Q_k] & [\nabla_x |V_k|] \end{bmatrix} \longrightarrow$$

$$[\nabla_x D_k] = \begin{bmatrix} \frac{\partial D_1}{\partial x_1} & \frac{\partial D_2}{\partial x_1} & \dots & \frac{\partial D_N}{\partial x_1} \\ \frac{\partial D_1}{\partial x_2} & \frac{\partial D_2}{\partial x_2} & \dots & \frac{\partial D_N}{\partial x_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial D_1}{\partial x_{nv}} & \frac{\partial D_2}{\partial x_{nv}} & \dots & \frac{\partial D_N}{\partial x_{nv}} \end{bmatrix}$$

where D_k can be P_k, Q_k, or $|V_k|$.

Hessian:

$$\nabla_x^2 g_i(\boldsymbol{x}) = \boldsymbol{H}_{g_i} = \left[\begin{array}{c|c|c} \frac{\partial^2 g_i}{\partial V_k^2} & \frac{\partial^2 g_i}{\partial V_k \partial P_k} = \boldsymbol{0} & \frac{\partial^2 g_i}{\partial V_k \partial Q_k} = \boldsymbol{0} \\ (u \times u) & (u \times n_g) & (u \times n_g) \\ \hline \frac{\partial^2 g_i}{\partial P_k \partial V_k} = \boldsymbol{0} & \frac{\partial^2 g_i}{\partial P_k^2} = \boldsymbol{0} & \frac{\partial^2 g_i}{\partial P_k \partial Q_k} = \boldsymbol{0} \\ (n_g \times u) & (n_g \times n_g) & (n_g \times n_g) \\ \hline \frac{\partial^2 g_i}{\partial Q_k \partial V_k} = \boldsymbol{0} & \frac{\partial^2 g_i}{\partial Q_k \partial P_k} = \boldsymbol{0} & \frac{\partial^2 g_i}{\partial Q_k^2} = \boldsymbol{0} \\ (n_g \times u) & (n_g \times n_g) & (n_g \times n_g) \end{array}\right]$$

Grad and Hessian of h(x)

All the inequality constraints are stored as

$$\boldsymbol{h}(\boldsymbol{x}) = \begin{bmatrix} P_{Gk}^{Min} \le P_{Gk} \le P_{Gk}^{Max} \\ Q_{Gk}^{Min} \le Q_{Gk} \le Q_{Gk}^{Max} \\ |V_k|^{Min} \le |V_k| \le |V_k|^{Max} \\ P_{km}^{Min} \le P_{km} \le P_{km}^{Max} \end{bmatrix}$$

$$\boldsymbol{h}(\boldsymbol{x}) = \begin{bmatrix} [P_{Gk}]^{Max} \\ [P_{Gk}]^{Min} \\ [Q_{Gk}]^{Max} \\ [Q_{Gk}]^{Min} \\ [|V_k|]^{Max} \\ [|V_k|]^{Min} \\ [P_{km}]^{Max} \\ [P_{km}]^{Min} \end{bmatrix} = \begin{bmatrix} P_{Gk} & - & P_{Gk}^{Max} \\ P_{Gk}^{Min} & - & P_{Gk} \\ Q_{Gk} & - & Q_{Gk}^{Max} \\ Q_{Gk}^{Min} & - & Q_{Gk} \\ |V_k| & - & |V_k|^{Max} \\ |V_k|^{Min} & - & |V_k| \\ P_{km} & - & P_{km}^{Max} \\ P_{km}^{Min} & - & P_{km} \end{bmatrix} \le \mathbf{0}$$

Line flow constraints

$$P_{km} = \frac{y_{km}^{Re}}{tap}\left[\left(\frac{(V_k^{Re})^2}{tap} + \frac{(V_k^{Im})^2}{tap}\right) - \left(V_k^{Re}V_m^{Re} + V_k^{Im}V_m^{Im}\right)\right] + \frac{y_{km}^{Im}}{tap}\left[\left(V_k^{Re}V_m^{Im} - V_k^{Im}V_m^{Re}\right)\right]$$

$$P_{km} = g_{km}|V_k|^2 - |V_k||V_m||y_{km}|\cos(\delta_k - \delta_m - \zeta_{km})$$

Grad:

$$\nabla_x \boldsymbol{h}(\boldsymbol{x}) = \begin{bmatrix} [\nabla_x P_G]^{Max} & [\nabla_x P_G]^{Min} & [\nabla_x Q_G]^{Max} & [\nabla_x Q_G]^{Min} \\ [\nabla_x |V_k|]^{Max} & [\nabla_x |V_k|]^{Min} & [\nabla_x P_{km}]^{Max} & [\nabla_x P_{km}]^{Min} \end{bmatrix}$$

$$[\nabla_x P_G]^{Max} = \begin{bmatrix} \frac{\partial [P_{G_k}]^{Max}}{\partial V_k} \\ \frac{\partial [P_{G_k}]^{Max}}{\partial P_{G_k}} \\ \frac{\partial [P_{G_k}]^{Max}}{\partial Q_{G_k}} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & (u \times n_g) \\ \mathbf{1} & (n_g \times n_g) \\ \mathbf{0} & (n_g \times n_g) \end{bmatrix} \qquad [\nabla_x Q_G]^{Max} = \begin{bmatrix} \frac{\partial [Q_{G_k}]^{Max}}{\partial V_k} \\ \frac{\partial [Q_{G_k}]^{Max}}{\partial P_{G_k}} \\ \frac{\partial [Q_{G_k}]^{Max}}{\partial Q_{G_k}} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & (u \times n_g) \\ \mathbf{0} & (n_g \times n_g) \\ \mathbf{1} & (n_g \times n_g) \end{bmatrix}$$

$$[\nabla_x |V_k|]^{Max} = \begin{bmatrix} \frac{\partial [|V_k|]^{Max}}{\partial V_k} \\ \frac{\partial [|V_k|]^{Max}}{\partial P_{G_k}} \\ \frac{\partial [|V_k|]^{Max}}{\partial Q_{G_k}} \end{bmatrix} = \begin{bmatrix} Nonzero & (u \times n_g) \\ \mathbf{0} & (n_g \times n_g) \\ \mathbf{0} & (n_g \times n_g) \end{bmatrix} \qquad [\nabla_x P_{km}]^{Max} = \begin{bmatrix} \frac{\partial [P_{km}]^{Max}}{\partial V_k} \\ \frac{\partial [P_{km}]^{Max}}{\partial P_{G_k}} \\ \frac{\partial [P_{km}]^{Max}}{\partial Q_{G_k}} \end{bmatrix} = \begin{bmatrix} Nonzero & (u \times n_g) \\ \mathbf{0} & (n_g \times n_g) \\ \mathbf{0} & (n_g \times n_g) \end{bmatrix}$$

$$[\nabla_x P_G]^{Min} = -[\nabla_x P_G]^{Max} \qquad [\nabla_x |V_k|]^{Min} = -[\nabla_x |V_k|]^{Max}$$

$$[\nabla_x Q_G]^{Min} = -[\nabla_x Q_G]^{Max} \qquad [\nabla_x P_{km}]^{Min} = -[\nabla_x P_{km}]^{Max}$$

Hessian:

$$\nabla_x^2 \boldsymbol{h}(\boldsymbol{x}) = \boldsymbol{H}_h = \begin{bmatrix} \frac{\partial^2 h_1}{\partial x^2} & \frac{\partial^2 h_2}{\partial x^2} & \cdots & \frac{\partial^2 h_{n_c}}{\partial x^2} \end{bmatrix}$$

$$\nabla_x^2 h_i(\boldsymbol{x}) = \boldsymbol{H}_{h_i} = \left[\begin{array}{c|c|c} \frac{\partial^2 h_i}{\partial V_k^2} & \frac{\partial^2 h_i}{\partial V_k \partial P_k} = \mathbf{0} & \frac{\partial^2 h_i}{\partial V_k \partial Q_k} = \mathbf{0} \\ (u \times u) & (u \times n_g) & (u \times n_g) \\ \hline \frac{\partial^2 h_i}{\partial P_k \partial V_k} = \mathbf{0} & \frac{\partial^2 h_i}{\partial P_k^2} = \mathbf{0} & \frac{\partial^2 h_i}{\partial P_k \partial Q_k} = \mathbf{0} \\ (n_g \times u) & (n_g \times n_g) & (n_g \times n_g) \\ \hline \frac{\partial^2 h_i}{\partial Q_k \partial V_k} = \mathbf{0} & \frac{\partial^2 h_i}{\partial Q_k \partial P_k} = \mathbf{0} & \frac{\partial^2 h_i}{\partial Q_k^2} = \mathbf{0} \\ (n_g \times u) & (n_g \times n_g) & (n_g \times n_g) \end{array}\right]$$

Thank you!